Kuwait University
Department of Mathematics and Computer Science

Math 102 | Date: 11/01/2005
Calculus II | Final Exam | Duration: 2 hours

Answer all questions. Calculators and Mobile Phones are not allowed.

1. (3 pts.) Given the function $f(x) = x^5 + x^3 + 2x - 2$.

(a) Show that f is one-to-one.

(b) Show that the point $P(2, 1)$ is on the graph of f^{-1}, and find the slope of the tangent line to the graph of f^{-1} at P.

2. (3 pts.) State whether each of the following statements is true or false – explain your answer.

(a) $\cos^{-1}[\cos(-\frac{\pi}{4})] = -\frac{\pi}{4}$

(b) $\cosh 4x - \sinh 4x = e^{-4x}$

(c) $\ln\sqrt{a+b} = \dfrac{1}{2}(\ln a + \ln b)$

3. (2 pts.) Find $\displaystyle\lim_{x\to 0^+} (1 + \csc x)^{\sin x}$

4. (5 pts. each) Evaluate the following integrals:

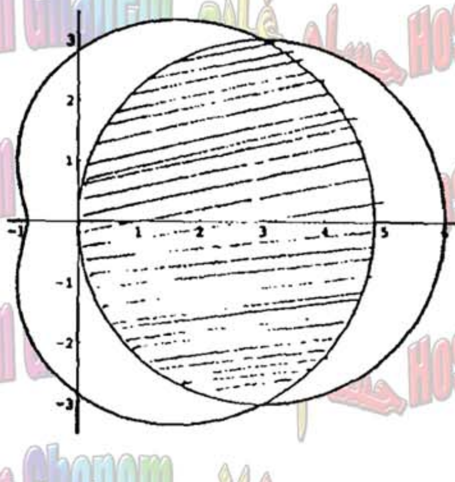
(a) $\displaystyle\int \frac{\tanh^3 x}{\sqrt{\operatorname{sech} x}}\, dx$

(b) $\displaystyle\int \frac{2x+3}{\sqrt{25 - x^2}}\, dx$

(c) $\displaystyle\int \tan^2 x \ \sec x \, dx$

(d) $\displaystyle\int \frac{x^2 + x - 5}{x^3 - 1}\, dx$

5. (5 pts.) Let $r_1 = 6\cos\theta$ and $r_2 = 2\sqrt{2} + 2\cos\theta$, where $0 \le \theta \le 2\pi$. Compute the shaded area.



6. (3 pts.) Find the length of the polar curve $r = \sin\theta + \cos\theta$, where $0 \le \theta \le 2\pi$.

7. (4 pts.) Given the equation

$$25x^2 + 9y^2 + 50x - 36y - 164 = 0,$$

(a) Identify the conic section represented by the above equation.

(b) Sketch the graph and find the center, vertices, and the foci.

Solutions of Final Exam, MATH 102, 11/01/2005

1a. $f'(x) = 5x^4 + 3x^2 + 2 > 0$ for all x implies f is increasing.
Thus, f is 1–1.

1b. $f(1) = 2$ implies $f^{-1}(2) = 1$ so $P(2,1)$ is on the graph of f^{-1}.
$f'(1) = 10$ implies $(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(1)} = \frac{1}{10}$

2a. [F] $\cos(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$ implies $\cos^{-1}[\frac{1}{\sqrt{2}}] = \frac{\pi}{4}$

2b. [T] $\frac{1}{2}[e^{4x} + e^{-4x} - e^{4x} + e^{-4x}] = e^{-4x}$

2c. [F] otherwise $\ln\sqrt{1+1} = 0$.

3. $\lim\limits_{y\to 0^+}(1+\csc x)^{\sin x} = 1$ by logarithmic diff and L'Hôpital's rule.

4a. $u = \operatorname{sech} x$, $\tanh x dx = -\frac{du}{u}$, $\tanh^2 x = 1 - u^2$. Thus, $\int \frac{\tanh^3 x}{\sqrt{\operatorname{sech} x}}\, dx =$
$-\int \frac{1-u^2}{u\sqrt{u}}\, du = 2u^{-1/2} + 2/3u^{3/2} + C = 2(\operatorname{sech} x)^{-1/2} + 2/3(\operatorname{sech} x)^{3/2} + C$

4b. $\int \frac{2x+3}{\sqrt{25-x^2}}\, dx = -2\sqrt{x^2+25} + 3\sin^{-1}\frac{x}{5} + C$

4c. Set $u = \tan x$, $dv = \sec x \tan x dx$ implies $du = \sec^2 x dx$, $v = \sec x$. Thus,

$$\begin{aligned}\int \tan^2 x \ \sec x\ dx &= \sec x \tan x - \int \sec^3 x dx \\ &= \sec x \tan x - \int (1+\tan^2 x)\sec x dx \\ &= \sec x \tan x - \ln|\sec x + \tan x| - \int \tan^2 x \sec x\end{aligned}$$

so that

$$\int \tan^2 x \sec x dx = \frac{1}{2}\Big[\sec x \tan x - \ln|\sec x + \tan x|\Big] + C$$

4d.

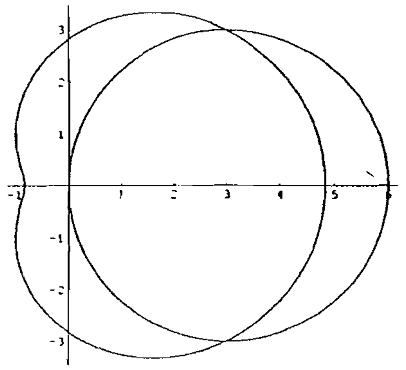
$$\frac{x^2+x-5}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

implies $A = -1$, $B = 2$ and $C = 4$. Hence,

$$\begin{aligned}\int \frac{x^2+x-5}{x^3-1}dx &= \int \frac{-1}{x-1}\, dx + \int \frac{2x+4}{x^2+x+1}\, dx \\ &= -\ln|x-1| + \ln|x^2+x+1| + 2\sqrt{3}\tan^{-1}\frac{2x+1}{\sqrt{3}} + C\end{aligned}$$

5. Put $r_1 = r_2$ to get $\cos\theta = \frac{1}{\sqrt{2}}$ which gives $\theta = \pm\pi/4$

$$\text{Area} \quad = 2\left[\int_0^{\pi/4} \frac{1}{2} r_2^2\, d\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2} r_1^2\, d\theta\right] = 2\left[-\frac{9}{2} + \frac{9\pi}{4} + \frac{9}{2} + \frac{5\pi}{4}\right] = 7\pi$$



6. $r' = \cos\theta - \sin\theta$. Hence, $(r')^2 + r^2 = 2$. Thus $L = \int_0^{2\pi} \sqrt{2}d\theta = 2\sqrt{2}\pi$.

7. After completing the square, the equation becomes $\frac{(y-2)^2}{25} + \frac{(x+1)^2}{9} = 1$, $a = 5, b = 3, c = 4$. The center: $(-1, 2)$, The vertices: $(-1, 2 \pm 5)$, the foci : $(-1, 2 \pm 4)$, The endpoints: $(-1 \pm 3, 2)$